

فصل هشتم

روش های

انتگرال گیری

تمرین صفحه 436 .

انتگرال $\int x^n \ln x \, dx$ را حل کنید.

حل) اگر فرض کنیم. $dv = x^n dx, u = \ln x$ داریم:

$$v = \frac{x^{n+1}}{n+1}, \quad du = \frac{dx}{x}$$

$$I_n = \frac{x^{n+1}}{n+1} \ln x - \int \frac{x^n}{n+1} dx = \frac{x^{n+1}}{n+1} \ln x - \frac{x^{n+1}}{(n+1)^2} + c \quad \text{پس}$$

تمرین صفحه 436 .

یک فرمول بازگشتی برای $I_n = \int \cos^n x \, dx$ ($n > 1$) پیدا

کنید و به کمک آن $\int \cos^4 x \, dx$ را محاسبه کنید.

حل) I_n را به صورت $I_n = \int \cos^{n-1} x \cdot \cos x \, dx$ می نویسیم.

اگر فرض کنیم $dv = \cos x \, dx, u = \cos^{n-1} x$ داریم

$du = -(n-1) \cos^{n-2} x \cdot \sin x \, dx$ و در نتیجه:

$$I_n = \sin x \cdot \cos^{n-1} x + (n-1) \int \cos^{n-2} x \cdot \sin^2 x \, dx$$

$$\Rightarrow I_n = \sin x \cdot \cos^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$$

$$\Rightarrow n I_n = \sin x \cdot \cos^{n-1} x + (n-1) I_{n-2}$$

$$n=2 \rightarrow 2I_2 = \sin x \cdot \cos x + x$$

$$n=4 \rightarrow 4I_4 = \sin x \cdot \cos^3 x + 3I_2$$

$$\Rightarrow I_4 = \frac{1}{4} (\sin x \cdot \cos^3 x + \frac{3}{2} (\sin x \cdot \cos x + x))$$

تمرین صفحه ۴۳۹ .

(۱) هر یک از انتگرال های زیر را محاسبه کنید.

$$1) \int e^x (f(x) + f'(x)) dx = f(x) \cdot e^x + c$$

$$2) \int \frac{x e^x}{(1+x)^2} dx = \frac{e^x}{x+1} + c$$

$$3) \int \text{Ln}(a^2 + x^2) dx = x \text{Ln}(a^2 + x^2) - \int \frac{x}{a^2 + x^2} dx$$

$$\Rightarrow x \text{Ln}(a^2 + x^2) - \frac{1}{2} \text{Ln}(a^2 + x^2) + c$$

$$4) \int x^3 e^{-x^2} dx = -\frac{1}{2} \int u e^u du = -\frac{1}{2} (-x^2 - 1) e^{-x^2} + c$$

$$5) \int x^5 e^x dx = (x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120) e^x$$

$$6) I = \int \sin(\text{Ln}x) dx$$

$$u = \sin(\text{Ln}x) \quad , dv = dx \Rightarrow du = \frac{1}{x} \cos(\text{Ln}x) \quad , v = x$$

$$I = x \sin(\text{Ln}x) - \int \cos(\text{Ln}x) dx$$

$$u = \cos(\text{Ln}x) \quad , dv = dx \Rightarrow du = -\frac{1}{x} \sin(\text{Ln}x) \quad , v = x$$

$$I = x \sin(\text{Ln}x) - x \cos(\text{Ln}x) - I$$

$$I = \frac{x}{2} (\sin(\text{Ln}x) - \cos(\text{Ln}x)) + c$$

$$7) I = \int x \text{tg}^{-1}x dx$$

$$u = \text{tg}^{-1}x, dv = x dx \Rightarrow du = \frac{1}{1+x^2} dx \quad , \quad v = \frac{x^2}{2}$$

$$I = \frac{x^2}{2} \text{tg}^{-1}x - \frac{1}{2} \int \frac{x^2}{x^2+1} dx = \frac{x^2}{2} \text{tg}^{-1}x - \frac{1}{2}x + \frac{1}{2} \text{tg}^{-1}x + c$$

$$8) \int \sin^{-1} \sqrt{x} dx \quad t^2 = x \Rightarrow 2t dt = dx$$

$$I = 2 \int t \sin^{-1} t dt = t^2 \sin^{-1} t - \int \frac{t^2}{\sqrt{1-t^2}} dt$$

$$= t^2 \sin^{-1} t + \int \frac{1-t^2+1}{\sqrt{1-t^2}} dt$$

$$= t^2 \sin^{-1} t + \sin^{-1} t + \int \sqrt{1-t^2} dt$$

$$\int \sqrt{1-t^2} dt = \int \cos^2 q dq = \frac{1}{2}t + \frac{1}{4} \cos 2q$$

$$\Rightarrow I = x \sin^{-1} \sqrt{x} + \sin^{-1} \sqrt{x} + \frac{1}{2} \cos^{-1} \sqrt{x} + \frac{1}{4} \cos 2(\cos^{-1} \sqrt{x})$$

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$$9) \quad I = \int (x^3 + x) \operatorname{ch} x \, dx$$

$$I = (x^3 + x) \operatorname{sh} x - (3x^2 + 1) \operatorname{ch} x + 6x \operatorname{sh} x - 6 \operatorname{ch} x$$

$$10) \quad I = \int \operatorname{Ln}(x + \sqrt{1+x^2}) \, dx$$

$$u = \operatorname{Ln}(x + \sqrt{1+x^2}), \, dv = dx \Rightarrow du = \frac{1}{\sqrt{1+x^2}}, \, v = x$$

$$I = x \operatorname{Ln}(x + \sqrt{1+x^2}) - \int \frac{x}{\sqrt{1+x^2}} = x \operatorname{Ln}(x + \sqrt{1+x^2}) - \sqrt{1+x^2} + c$$

$$11) \quad I = \int x \operatorname{Ln}\left(\frac{1+x}{1-x}\right) dx$$

$$u = \operatorname{Ln}\left(\frac{1+x}{1-x}\right), \, dv = dx \Rightarrow du = \frac{2}{1-x^2} dx, \, v = x$$

$$I = x \operatorname{Ln}\left(\frac{1+x}{1-x}\right) - \int \frac{2x}{1-x^2} dx = x \operatorname{Ln}\left(\frac{1+x}{1-x}\right) + \operatorname{Ln}(1-x^2) + c$$

$$12) \quad I = \int x^2 \operatorname{Ln}\left(\frac{1+x}{1-x}\right) dx$$

$$u = \operatorname{Ln}\left(\frac{1-x}{1+x}\right), \quad dv = x^2 dx \Rightarrow du = -\frac{2}{1-x^2} dx, \quad v = \frac{x^3}{3} dx$$

$$I = \frac{x^3}{3} \operatorname{Ln}\left(\frac{1-x}{1+x}\right) + \frac{1}{3} \int x^3 (1-x^2) dx$$

$$= \frac{x^3}{3} \operatorname{Ln}\left(\frac{1-x}{1+x}\right) - \frac{1}{3} \int \left(x + \frac{x}{x^2-1}\right) dx$$

$$= \frac{x^3}{3} \operatorname{Ln}\left(\frac{1-x}{1+x}\right) - \frac{1}{3} x^2 - \frac{1}{6} \operatorname{Ln}(x^2-1) + c$$

$$13) \quad I = \int e^{\sqrt{x}} dx$$

$$u^2 = x \Rightarrow 2u du = dx$$

$$I = 2 \int u e^u du = 2(u-1)e^u + c = 2(\sqrt{x}-1)e^{\sqrt{x}} + c$$

$$14) \quad I = \int \frac{\operatorname{tg}^{-1} e^x}{e^x} dx$$

$$u = \operatorname{tg}^{-1} e^x, \quad dv = \frac{dx}{e^x} \Rightarrow du = \frac{e^x}{1+e^{2x}} dx, \quad v = -\frac{1}{e^x}$$

$$I = -\frac{\operatorname{tg}^{-1} e^x}{e^x} + \int \frac{dx}{1+e^{2x}} = \frac{\operatorname{tg}^{-1} e^x}{e^x} + \int \frac{du}{u(1+u^2)}$$

$$= -\frac{\operatorname{tg}^{-1} e^x}{e^x} + \operatorname{Ln} e^x + \frac{1}{2} \operatorname{Ln}(1+e^{2x}) - 2 \operatorname{tg}^{-1} e^x + c$$

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$$15) \quad I = \int (\sin^{-1} x)^2 dx$$

$$u = (\sin^{-1} x)^2, \quad dv = dx \quad \Rightarrow \quad du = \frac{2}{\sqrt{1-x^2}} \sin^{-1} x dx, \quad v = x$$

$$\begin{aligned} I &= x(\sin^{-1} x)^2 - \int \frac{2}{\sqrt{1-x^2}} \sin^{-1} x dx \\ &= x(\sin^{-1} x)^2 + 2(\sqrt{1-x^2}) \sin^{-1} x - 2x + c \end{aligned}$$

$$16) \quad I = \int \left(\frac{\operatorname{Ln} x}{x}\right)^2 dx$$

$$u = \left(\frac{\operatorname{Ln} x}{x}\right)^2, \quad dv = dx \quad \Rightarrow \quad du = 2\left(\frac{1-\operatorname{Ln} x}{x^2}\right)\left(\frac{\operatorname{Ln} x}{x}\right) dx, \quad v = x$$

$$I = x\left(\frac{\operatorname{Ln} x}{x}\right)^2 - 2 \int \frac{\operatorname{Ln} x - (\operatorname{Ln} x)^2}{x^2} dx$$

$$= x\left(\frac{\operatorname{Ln} x}{x}\right)^2 - 2 \int \frac{\operatorname{Ln} x}{x^2} dx + 2 \int \left(\frac{\operatorname{Ln} x}{x}\right)^2 dx$$

$$-I = x\left(\frac{\operatorname{Ln} x}{x}\right)^2 + 2 \int \frac{1-\operatorname{Ln} x-1}{x^2} dx$$

$$I = -x\left(\frac{\operatorname{Ln} x}{x}\right)^2 - 2\left(\int \left(\frac{\operatorname{Ln} x}{x}\right)' dx - \int \frac{1}{x^2} dx\right) + c$$

$$= -x\left(\frac{\operatorname{Ln} x}{x}\right)^2 - 2\left(\frac{\operatorname{Ln} x}{x}\right) + \frac{1}{x} + c$$

$$17) \quad I = \int \frac{\operatorname{Ln}(1+x)}{2(1+x)} dx = \frac{1}{2} \int u du$$

$$= \frac{1}{4} (\operatorname{Ln}(1+x))^2 + c$$

$$18) \quad I = \int_0^p \frac{p^2}{2} \cos \sqrt{2x} dx$$

$$u = \sqrt{2x} \quad \Rightarrow \quad 2u du = 2dx$$

$$I = \int_0^p u \cos u du = u \sin u + \cos u + c$$

$$19) \quad I = \int_1^4 \sec^{-1} \sqrt{x} dx, \quad u = \sqrt{x}$$

$$2u du = dx$$

$$I = 2 \int_1^2 u \sec^{-1} u du = 2 \int_1^2 u \cos^{-1} \left(\frac{1}{u}\right) du$$

$$t = \cos^{-1} \left(\frac{1}{u}\right) \Rightarrow dt = -\frac{1}{u^2} \cdot \frac{-1}{\sqrt{1-\frac{1}{u^2}}} du = \frac{1}{u\sqrt{u^2-1}} du$$

$$I = 2\left(\frac{u^2}{2} \cos^{-1}\left(\frac{1}{u}\right)\right) - \int \frac{u}{\sqrt{u^2-1}} du$$

$$= u^2 \cos^{-1}\left(\frac{1}{u}\right) - \sqrt{u^2-1} \Big|_1^2 = 4 \times \frac{p}{3} - \sqrt{3}$$

$$20) \quad I = \int_{\frac{p}{4}}^{\frac{3p}{4}} x \cdot \cot x \cdot \csc x \, dx = - \int_{\frac{p}{4}}^{\frac{3p}{4}} x (\csc x)' \, dx$$

$$u = x, \quad dv = (\csc x)' \, dx \Rightarrow du = dx, \quad v = \csc x$$

$$I = x \csc x \Big|_{\frac{p}{4}}^{\frac{3p}{4}} - \int_{\frac{p}{4}}^{\frac{3p}{4}} \csc x \, dx$$

$$\int \csc x \, dx = \int \frac{dx}{\sin x} = \int \frac{\sin x}{1 - \cos^2 x} \, dx = \frac{1}{2} \operatorname{Ln} \left| \frac{1 - \cos x}{1 + \cos x} \right|$$

$$I = \left(\frac{3p}{4} \times \sqrt{2} - \frac{p}{4} \times \sqrt{2}\right) - \frac{1}{2} \left(\operatorname{Ln} \left| \frac{\sqrt{2}+1}{\sqrt{2}-1} \right| - \operatorname{Ln} \left| \frac{\sqrt{2}-1}{\sqrt{2}+1} \right| \right)$$

$$21) \quad I = \int_0^{\frac{p^2}{4}} \sin \sqrt{x} \, dx \quad u^2 = x \Rightarrow 2u \, du = dx$$

$$I = 2 \int_0^{\frac{p}{2}} u \sin u \, du = 2(-u \cos u + \sin u) \Big|_0^{\frac{p}{2}}$$

تمرین صفحه ۴۴۴ .

هر یک از انتگرال های زیر را حل کنید.

$$1) \int \frac{x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4}{(x^2 + 2)^3} dx$$

$$f(x) = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2} + \frac{Ex + F}{(x^2 + 1)^2}$$

$$\begin{aligned} \Rightarrow (Ax + B)(x^4 + 2x^2 + 1) + (Cx + D)(x^2 + 1) + Ex + F \\ = x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4 \end{aligned}$$

$$\Rightarrow A = 1, \quad B = -1, \quad 2A + C = 4 \Rightarrow C = 2$$

$$2B$$

$$2) \int \frac{(\sec^2 x + 1) \cdot \sec^2 x}{1 + \tan^2 x}$$

$$u = \tan x \Rightarrow du = \sec^2 x dx, \quad u^2 = \sec^2 x - 1$$

$$I = \int \frac{(u^2 + 2)du}{1 + u^3}$$

$$\frac{u^2 + 2}{1 + u^3} = \frac{A}{1 + u} + \frac{Bu + C}{1 + u + u^2}$$

$$\Rightarrow (A + B)u^2 + (A + B + C)u + A + C = u^2 + 2$$

$$A + B = 1$$

$$A + B + C = 0 \Rightarrow C = -1, \quad A = 3, \quad B = -2$$

$$A + C = 2$$

$$\Rightarrow I = \int \frac{3}{u + 1} du - \int \frac{2u + 1}{u^2 + u + 1} du = 3 \ln(u + 1) - \ln(u^2 + u + 1) + c$$

$$I = 3 \ln(\tan x + 1) - \ln(\tan^2 x + \tan x + 1) + c$$

$$3) \quad I = \int \frac{x^2 + 2x - 1}{27x^3 - 1} dx$$

$$\frac{x^2 + 2x - 1}{(3x - 1)(9x^2 + 3x + 1)} = \frac{A}{3x - 1} + \frac{Bx + C}{9x^2 + 3x + 1}$$

$$\Rightarrow (9A + 3B)x^2 + (3A - B + 3C)x + A = x^2 + 2x - 1$$

$$\Rightarrow 9A + 3B = 1 \quad , \quad 3A - B + 3C = 2 \quad , \quad A - C = -1$$

$$A = C - 1 \quad , \quad 9C - 9 + 3B = 1$$

$$3C - 3 - B + 3C = 2$$

$$\Rightarrow \begin{cases} 9C + 3B = 10 \\ 6C - B = 5 \end{cases} \Rightarrow 27C = 25 \Rightarrow C = \frac{25}{27}$$

$$A = \frac{-2}{27} \quad , \quad B = \frac{75}{9} - \frac{45}{9} = \frac{10}{3}$$

$$\Rightarrow -\frac{2}{27} \int \frac{dx}{3x - 1} = -\frac{2}{81} \ln(3x - 1)$$

$$\frac{\frac{10}{3}x - \frac{25}{27}}{9x^2 + 3x + 1} = \frac{1}{27} \cdot \frac{90x - 25}{9x^2 + 3x + 1}$$

$$I = -\frac{2}{81} \ln(3x - 1) + \frac{1}{27} \int \frac{90x - 25}{9x^2 + 3x + 1} dx$$

$$4) \quad I = \int \frac{dx}{x^3 + x^2 + x} = \int \frac{dx}{x(x^2 + x + 1)}$$

$$\frac{1}{x(x^2 + x + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + x + 1}$$

$$A = 1 \quad , \quad B = -1 \quad , \quad C = -1$$

$$I = \int \frac{dx}{x} - \int \frac{x + 1}{x^2 + x + 1} = \ln x - \frac{1}{2} \ln(x^2 + x + 1) + \frac{2}{\sqrt{3}} \operatorname{tg}^{-1} \left(\frac{x + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right)$$

$$5) \int \frac{dq}{\cos q(1 + \sin q)}$$

$$u = \sin q \Rightarrow \frac{du}{\cos q} = dq$$

$$I = \int \frac{du}{(1-u)^2(1+u)} = \int \frac{du}{(1+u)^2(1-u)}$$

$$\frac{1}{(1+u)^2(1-u)} = \frac{A}{1+u} + \frac{B}{(1+u)^2} + \frac{C}{1-u}$$

$$C = \frac{1}{4}, \quad B = \frac{1}{2}, \quad A = \frac{1}{4}$$

$$I = \frac{1}{4} \int \frac{du}{1+u} + \frac{1}{2} \int \frac{du}{(1+u)^2} - \frac{1}{4} \int \frac{du}{u-1}$$

$$I = \frac{1}{4} \ln(1 + \sin q) - \frac{1}{2(1 + \sin q)} - \frac{1}{4} \ln(\sin q - 1) + C$$

$$6) \int \frac{dq}{\sin(1 + \sin q)} = \int \frac{dq}{\sin q} - \int \frac{dq}{1 + \sin q}$$

$$= \frac{1}{2} \ln \left| \frac{\sin q - 1}{\sin q + 1} \right| - \int \frac{1 - \sin q}{\cos^2 q} dq$$

$$= \frac{1}{2} \ln \left| \frac{\sin q - 1}{\sin q + 1} \right| - \operatorname{tg} q = \frac{1}{\cos q} + C$$

$$7) \int \frac{dt}{(1+t)(1+t^2)^2}$$

$$\frac{1}{(1+t)(1+t^2)^2} = \frac{A}{1+t} + \frac{Bt+C}{t^2+1} + \frac{Dt+E}{(t^2+1)^2}$$

$$A(t^2+1)^2 + (Bt+C)(t^2+1) + dt + E = 1$$

$$A = \frac{1}{4}, \quad B = -\frac{1}{4}$$

$$A + C + E = 1 \Rightarrow C + E = \frac{3}{4}$$

$$8) \int \frac{x^2+1}{x^3+8} dx$$

$$\frac{x^2+1}{x^3+8} = \frac{x^2+1}{(x^2+2)(x^2+2x+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+2x+4}$$

$$\Rightarrow (A+B)x^2 + (2A+2B+2C)x + 4A+2C = x^2+1$$

$$A+B=1$$

$$\Rightarrow C = -1, \quad A = \frac{3}{4}, \quad B = \frac{1}{4}$$

$$A+B+C=0$$

$$4A+2C=1$$

$$I = \frac{1}{4} \int \frac{dx}{x+2} + \frac{1}{4} \int \frac{3x+1}{x^2+2x+4} dx$$

$$= \frac{1}{4} \ln(x+2) + \frac{1}{4} \int \frac{2x+2+(x-1)}{x^2+2x+4} dx$$

$$= \frac{1}{4} \ln(x+2) + \frac{1}{4} \ln(x^2+2x+4) + \frac{1}{4} \int \frac{(x-1)}{x^2+2x+4}$$

$$\int \frac{(x-1)}{(x+1)^2+3} dx = \frac{1}{2} \ln((x+1)^2+3) - \frac{2}{\sqrt{3}} \operatorname{tg}^{-1}\left(\frac{x+1}{\sqrt{3}}\right)$$

$$9) \int \frac{dx}{x^4 - 3x^3} = \int \frac{dx}{x^3(x-3)}$$

$$\frac{dx}{x^3(x-3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-3}$$

$$C = -\frac{1}{3}, \quad D = 1, \quad A = -1, \quad B = \frac{4}{3}$$

$$I = -\ln x - \frac{4}{3} \cdot \frac{1}{x} - \frac{1}{6} \cdot \frac{1}{x^2} + \ln(x-3) + C$$

$$10) \int \frac{x^2 + 2}{4x^5 + 4x^3 + x} = \int \frac{x^2 + 2}{x(2x^2 + 1)^2}$$

$$\frac{x^2 + 2}{x(2x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{2x^2 + 1} + \frac{Dx + E}{(2x^2 + 1)^2}$$

$$A = 2, \quad B = -4, \quad \dots$$

$$11) \int \frac{xdx}{(x^2 - x + 1)^2}$$

$$\frac{x}{(x^2 - x + 1)^2} = \frac{Ax + B}{x^2 - x + 1} = \frac{Cx + D}{(x^2 - x + 1)^2}$$

$$A = 1, \quad B + D = 0, \quad C = 0$$

$$3B + D = 2 \Rightarrow B = 1, \quad D = -1$$

$$12) \int_0^1 \frac{dx}{x^3 + 2x^2 + x + 2}$$

$$13) \int_0^4 \frac{x^2 dx}{2x^3 + 9x^2 + 12x + 4}$$

$$14) \int_{-1}^0 \frac{x^2 dx}{(2x^2 + 2x + 1)^2}$$

تمرین صفحه 449.

هر یک از انتگرال های زیر را حل کنید.

$$1) \int \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2} \sin^{-1}(2x) + C$$

$$2) \int \frac{x^2 dx}{\sqrt{9-x^2}}$$

$$x = 3 \sin q \Rightarrow dx = 3 \cos q dq$$

$$I = \int \frac{9 \sin^2 q \cdot 3 \cos q dq}{3 \cos q} = \frac{9}{2} q - \frac{9}{4} \sin 2q$$

$$3) \int \frac{x^3 dx}{\sqrt{9-x^2}} = 27 \int \sin^3 q dq = 27 \int \sin q - \cos^2 q \sin q$$

$$= -27 \cos q + 9 \cos^3 q + c$$

$$4) \int \frac{dx}{x\sqrt{9-x^2}} = \frac{1}{3} \int \frac{dq}{\sin q} = -\frac{1}{6} \ln \left| \frac{\sin q + 1}{\sin q - 1} \right| + 2$$

$$5) \int x^2 \sqrt{9-x^2} dx$$

$$x = 3 \sin q \Rightarrow I = 81 \int \sin^2 q \cdot \cos^2 q dq$$

$$I = \frac{81}{4} \int \sin^2 2q dq = \frac{81}{8} q - \frac{81}{32} \sin 4q + C$$

$$6) \int \frac{x+1}{\sqrt{9-x^2}} dx = \int (\sin q + 1) dq = -\cos q + q$$

$$7) \int \frac{x^3}{\sqrt{9+x^2}} dx \quad x = 3 \operatorname{tg} q \Rightarrow dx = 3 \sec^2 q dq$$

$$I = 27 \int \operatorname{tg}^3 q dq = 27 \int \operatorname{tg} q (\sec^2 q - 1) dq$$

$$= \frac{27}{2} \operatorname{tg}^2 q + 27 \ln(\cos q) + C$$

$$8) \quad I = \int \frac{\sqrt{1-x^2}}{x^2} dx = \int \frac{\cos^2 q}{\sin^2 q} dq = -\cot q - q + C$$

$$9) \quad I = \int_{-\ln 2}^0 e^x \sqrt{1-e^{2x}} dx$$

$$u = e^x \Rightarrow I = \int_{\frac{1}{2}}^1 \sqrt{1-u^2} du = \int_{\frac{p}{6}}^{\frac{p}{2}} \cos^2 q dq$$

$$= \left(\frac{1}{2}q + \frac{1}{4} \sin 2q \right) \Big|_{\frac{p}{6}}^{\frac{p}{2}} = \frac{p}{4} - \left(\frac{p}{12} + \frac{\sqrt{3}}{8} \right)$$

$$10) \quad \int \frac{dx}{(x^2+2x+2)^2} = \int \frac{dx}{((x+1)^2+1)^2} = \int \frac{du}{(u^2+1)^2}$$

$$\int \frac{\sec^2 q}{\sec^4 q} dq = \int \cos^2 q dq = \frac{1}{2}q + \frac{1}{4} \sin 2q + C$$

$$11) \quad \int \frac{dx}{(1+2x^2)^{\frac{5}{2}}}, \quad \sqrt{2}x = \operatorname{tg} q$$

$$I = \frac{1}{\sqrt{2}} \int \frac{\sec^2 q}{\sec^5 q} dq = \frac{1}{\sqrt{2}} \int \cos^3 q dq$$

$$= \frac{1}{\sqrt{2}} \int (\cos q - \sin^2 q \cos q) dq = \frac{1}{\sqrt{2}} \sin q - \frac{\sin^3 q}{3\sqrt{2}} + C$$

$$\begin{aligned}
 12) \quad \int \frac{dq}{2 + \sin q} &= \int \frac{\frac{2dt}{1+t^2}}{2 + \frac{2t}{1+t^2}} dt = \int \frac{dt}{t^2 + t + 1} \\
 &= \frac{2}{\sqrt{3}} \operatorname{tg}^{-1} \left(\frac{t + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) + C
 \end{aligned}$$

$$\begin{aligned}
 13) \quad \int_0^{\frac{p}{2}} \frac{dq}{1 + \sin q + \cos q} &= \int_0^1 \frac{\frac{2dt}{1+t^2}}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} = \int_0^1 \frac{2dt}{2+2t} \\
 &= \ln(1+t) \Big|_0^1 = \ln 2
 \end{aligned}$$

$$\begin{aligned}
 14) \quad \int \frac{dq}{3 + 2\cos q} &= \int \frac{\frac{2dt}{1+t^2}}{3 + 2 + \frac{1-t^2}{1+t^2}} dt \\
 &= \int \frac{2dt}{t^2 + 5} = \frac{2}{\sqrt{5}} \operatorname{tg}^{-1} \left(\frac{t}{\sqrt{5}} \right) + C
 \end{aligned}$$

$$15) \int \sec^3 x dx = \int \frac{dx}{\cos^2 x} = \int \frac{du}{(1-u^2)^2}$$

$$u = \sin x \Rightarrow dx = \frac{du}{\cos x}$$

$$\frac{1}{(1-u)^2(1+u)^2} = \frac{A}{u+1} + \frac{B}{(u+1)^2} + \frac{C}{u-1} + \frac{D}{(u-1)^2}$$

$$D = \frac{1}{4}, \quad B = \frac{1}{4}, \quad A + C = 0, \quad A - C = -\frac{1}{2}$$

$$A = -\frac{1}{4}, \quad C = \frac{1}{4}$$

$$I = -\frac{1}{4} \ln|\sin x + 1| - \frac{1}{2} \cdot \frac{1}{1 + \sin x} + \frac{1}{4} \ln|\sin x - 1| - \frac{1}{4} \cdot \frac{1}{\sin x - 1}$$

$$16) \int \csc x dx = \int \frac{dx}{\sin x} = \int \frac{dt}{t} = \ln \left| \operatorname{tg} \frac{x}{2} \right| + C$$

$$17) I = \int x \sin^{-1} x dx = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} dx$$

$$= \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \int \sin^2 q dq$$

$$= \frac{x^2}{2} \sin^{-1} x - \frac{1}{4} (\sin^{-1} x) + \frac{1}{8} \sin 2(\sin^{-1} x) + C$$

$$18) \int \frac{dx}{1 - \sin x + \cos x} = \int \frac{\frac{2dt}{1+t^2}}{1 - \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} = \int \frac{dt}{1-t}$$

$$= -\ln \left| \operatorname{tg} \frac{x}{2} - 1 \right| + C$$

$$19) \int_0^p \frac{\sin x}{4 + \cos^2 x} dx = -\int_1^{-1} \frac{du}{4 + u^2} = \frac{1}{2} \operatorname{tg}^{-1} u \Big|_{-1}^1 = \frac{1}{2} \times \frac{p}{2} = \frac{p}{4}$$

$$20) \int \frac{dx}{x^4 \sqrt{x^2-1}} = \int \frac{\sec q \operatorname{tg} q}{\sec^4 q \operatorname{tg} q} dq = \int \cos^3 q dq$$

$$= \int (1 - \sin^2 q) \cos q dq = \sin q - \frac{\sin^3 q}{3} + C$$

$$21) \int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(x-1)^2}} = \sin^{-1}(x-1) + C$$

$$22) \int \frac{dx}{4x^2 + 12x + 13} = \int \frac{dx}{(2x+3)^2 + 4} = \frac{1}{4} \operatorname{tg}^{-1} \left(\frac{2x+3}{2} \right)$$